Temporal Logic: Support de cours de DEA 1995/96

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In this chapter, we present temporal logics using modal type operators. When we want to describe a temporal relation, we first will fix the properties this relation should have. For example, we might want that the relation be linear or transitive. According to possible properties of the binary tense relation, different axiomatics may be defined. In this chapter, we first present in detail the "smallest" temporal system, we can think of: the temporal relation does not have any particular property. Subsequently, we present some results on temporal systems where the time has other properties.

1 The language of temporal logic

The language of temporal logic, $\mathcal{L}$, is an extension of the language of classical predicate logic. $\mathcal{L}$ consists of

- individual variables x, y,...
- ω-many m_i - place function symbols, F_i for $i \in \omega$
- ω-many m_j - place predicate symbols P_j for $j \in \omega$
- the logical symbols $\neg$, $\vee$,$\exists$
- the tense operators **F**, **P**, **G**, **H**

Terms and formulas are defined as usual, and if ϕ is a formula then so are $\mathbf{F}\phi$, $\mathbf{G}\phi$, $\mathbf{H}\phi$ and $\mathbf{P}\phi$. $\wedge$, $\rightarrow$, $\forall$ are defined as in classical logic.

F and **P** are defined in terms of **G** and **H**:

$\mathbf{F}\phi \leftrightarrow \neg\mathbf{G}\neg\phi$ and $\mathbf{P}\phi \leftrightarrow \neg\mathbf{H}\neg\phi$

If ϕ is a formula then we denote by $\phi_x[c]$ the formula obtained from ϕ by replacing every free occurrence of x by c. If c is the name of an element of a set $\mathcal{O}$ then $\phi_x[c]$ is called $\mathcal{O}$-instance of ϕ. In order to omit parentheses, we assume that the one-place operators bind closer that the two place operators and $\wedge$, $\vee$ bind closer than $\rightarrow$, (i.e. $\phi \wedge \psi \rightarrow \rho \vee \sigma$ is $(\phi \wedge \psi) \rightarrow (\rho \vee \sigma)$, and a sequence of $\rightarrow$ is parenthesized to the right (i.e. $\phi \rightarrow \psi \rightarrow \rho \rightarrow \sigma$ is $\phi \rightarrow (\psi \rightarrow (\rho \rightarrow \sigma))$. We call atomic formula or atom any positive literal containing no free variables (i.e. containing only constants or terms containing only constants).

2 Axioms and inference rules of temporal logic

According to the properties one wants to give to a before-after relation (e.g. transitiveness, antisymmetry, linearity, etc.) there are different axiom systems for temporal logic. We will subsequently give the axioms of the system $\mathcal{K}_t$ of minimal tense logic. Minimal means here that the temporal relation has no special property, the set of axioms for $\mathcal{K}_t$ will be shown to be true in any temporal structure.

Axiom 1 *all valid formulas of classical predicate logic*

Axiom 2 *(a)* $\mathbf{G}(\phi \rightarrow \psi) \rightarrow \mathbf{G}\phi \rightarrow \mathbf{G}\psi$ *(b)* $\mathbf{H}(\phi \rightarrow \psi) \rightarrow \mathbf{H}\phi \rightarrow \mathbf{H}\psi$

Axiom 3 *(a)* $\phi \rightarrow \mathbf{G}\neg\mathbf{H}\neg\phi$ *(b)* $\phi \rightarrow \mathbf{H}\neg\mathbf{G}\neg\phi$

Rule 1 *all valid inference rules of classical predicate logic*

Rule 2 *(a)* $\mathbf{G}$ *- introduction :* $\phi : \mathbf{G}\phi$ *(b)* $\mathbf{H}$ *- introduction:* $\phi : \mathbf{H}\phi$

We write $\vdash_{\mathcal{K}_t} \phi$ (or $\vdash \phi$, when no confusion can occur) if ϕ is a theorem of $\mathcal{K}_t$. When we introduce a set of non-logical axioms $\mathcal{M}$ from which ϕ can be deduced then we write $\mathcal{M} \vdash \phi$.

One might object that $\mathcal{K}_t$ does not include axioms $\mathbf{G}\phi \rightarrow \neg\mathbf{G}\neg\phi$ and $\mathbf{H}\phi \rightarrow \neg\mathbf{H}\neg\phi$ which intuitively appear to be true. But $\mathbf{G}\phi \rightarrow \neg\mathbf{G}\neg\phi$ (resp. $\mathbf{H}\phi \rightarrow \neg\mathbf{H}\neg\phi$) is not true in a tense structure which has ending (resp. beginning) points.

3 Theorems and derivation rules which can be proven from the above calculus

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G and **H** are dual operators. Therefore all theorems and deductions rules which are proven for one hold also for the other. Consequently, we shall always elaborate only one of the proofs of theorems which are true dually for **G** and **H**.

D1 from $\phi \rightarrow \psi$ infer

1.a $\mathbf{G}\phi \rightarrow \mathbf{G}\psi$ b. $\mathbf{H}\phi \rightarrow \mathbf{H}\psi$

2.a $\mathbf{F}\phi \rightarrow \mathbf{F}\psi$ b. $\mathbf{P}\phi \rightarrow \mathbf{P}\psi$

Proof of 1.a:

(1) $\mathbf{G}(\phi \rightarrow \psi)$ by R1

(2) $\mathbf{G}\phi \rightarrow \mathbf{G}\psi$ from (1) by axiom 1 and R0

Proof of 2.b:

(1) $\neg\psi \rightarrow \neg\phi$

(2) $\mathbf{H}\neg\psi \rightarrow \mathbf{H}\neg\phi$ by D1 a

(3) $\neg\mathbf{H}\neg\phi \rightarrow \neg\mathbf{H}\neg\psi$ by R0.

D2 from $\phi \rightarrow \psi$ infer $\heartsuit\phi \rightarrow \heartsuit\psi$, where $\heartsuit$ is any sequence of temporal operators. Proof : immediately from D1.

T1 a $\mathbf{G}(\phi \wedge \psi) \leftrightarrow \mathbf{G}\phi \wedge \mathbf{G}\psi$ b. $\mathbf{H}(\phi \wedge \psi) \leftrightarrow \mathbf{H}\phi \wedge \mathbf{H}\psi$

Proof :

(1) $\phi \wedge \psi \rightarrow \phi$ by AO

(2) $\mathbf{G}(\phi \wedge \psi) \rightarrow \mathbf{G}\phi$ by D1

(3) $\mathbf{G}(\phi \wedge \psi) \rightarrow \mathbf{G}\psi$ analogous (2)

(4) $\mathbf{G}(\phi \wedge \psi) \rightarrow \mathbf{G}\phi \wedge \mathbf{G}\psi$ (2) and (3)

(5) $\phi \rightarrow (\psi \rightarrow \phi \wedge \psi)$ AO

(6) $\mathbf{G}\phi \rightarrow \mathbf{G}(\psi \rightarrow \phi \wedge \psi)$ by D1

(7) $\mathbf{G}(\psi \rightarrow \phi \wedge \psi) \rightarrow \mathbf{G}\psi \rightarrow \mathbf{G}(\phi \wedge \psi)$

(8) $\mathbf{G}\phi \wedge \mathbf{G}\psi \rightarrow \mathbf{G}(\phi \wedge \psi)$ (6) and (7)

T2 a $\mathbf{F}(\phi \wedge \psi) \rightarrow \mathbf{F}\phi$ b. $\mathbf{P}(\phi \wedge \psi) \rightarrow \mathbf{P}\phi$

Proof : by AO, D1. 2.

T3 a $\mathbf{FH}\phi \rightarrow \phi$ b. $\mathbf{PG}\phi \rightarrow \phi$

Proof directly from A2.

T4 a $\mathbf{G}(\phi \rightarrow \psi) \wedge \mathbf{F}\phi \rightarrow \mathbf{F}\psi$ b. $\mathbf{H}(\phi \rightarrow \psi) \wedge \mathbf{P}\phi \rightarrow \mathbf{P}\psi$

Proof :

(1) $\neg\psi \wedge (\neg\psi \rightarrow \neg\phi) \rightarrow \neg\phi$

(2) $\mathbf{G}(\neg\psi \wedge (\neg\psi \rightarrow \neg\phi)) \rightarrow \mathbf{G}\neg\phi$ from (1) by D1.a.

(3) $\mathbf{G}\neg\psi \wedge \mathbf{G}(\neg\psi \rightarrow \neg\phi) \rightarrow \mathbf{G}\neg\phi$ from (2) by T1.a.

(4) $\mathbf{G}(\neg\psi \rightarrow \neg\phi) \wedge \neg\mathbf{G}\neg\phi \rightarrow \neg\mathbf{G}\neg\psi$ from (3) by R0.

(5) $\mathbf{G}(\phi \rightarrow \psi) \wedge \mathbf{F}\phi \rightarrow \mathbf{F}\psi$ from (4) by contraposition.

T5 a $\mathbf{F}(\phi \rightarrow \psi) \wedge \mathbf{G}\phi \rightarrow \mathbf{F}\psi$ b.$\mathbf{P}(\phi \rightarrow \psi) \wedge \mathbf{H}\phi \rightarrow \mathbf{P}\psi$

Proof :

(1) $\phi \wedge \neg\psi \rightarrow \neg(\phi \rightarrow \psi)$

(2) $\mathbf{G}(\phi \wedge \neg\psi) \rightarrow \mathbf{G}\neg(\phi \rightarrow \psi)$ from (1) by D1.a.

(3) $\mathbf{G}\phi \wedge \mathbf{G}\neg\psi \rightarrow \mathbf{G}\neg(\phi \rightarrow \psi)$ from (2) by T1.a.

(4) $\neg\mathbf{G}\neg(\phi \rightarrow \psi) \wedge \mathbf{G}\phi \rightarrow \neg\mathbf{G}\neg\psi$ from (3) by R0.

Furthermore, we shall often use theorems or derivation rules proven in classical predicate logic as in [6], for example.

4 Semantical characterization of tense logic

Tense structures for $\mathcal{K}_t$ ($\mathcal{K}_t$- structures) are Kripke-type structures [2], i.e. sets of classical structures.

Definition 1 *The pair* $(\{\mathcal{A}_s : s \in \mathcal{S}\}, \mathcal{R})$ *is called tense-structure, if*

- $\mathcal{S} \neq \emptyset$ *is the set of states*
- $R \subseteq \mathcal{S} \times \mathcal{S}$ *is a binary relation on* $\mathcal{S}$*, called temporal relation*
- *for every* $s \in \mathcal{S}$, $\mathcal{A}_s$ *is a classical structure* $\mathcal{A}_s = (\mathcal{O}, (f_i^s)_{i\in\omega}, (p_j^s)_{j\in\omega}, =)$, ω *being the set of natural numbers.*
  - $\mathcal{O} \neq \emptyset$ *is the set of objects*
  - $f_i^s : \mathcal{O}^{n_i} \rightarrow \mathcal{O}$ *is an* n_i*-place function on* $\mathcal{O}$*, for* $n_i \in \omega$
  - $p_j^s \subseteq \mathcal{O}^{m_j}$ *is an* m_j*-place predicate on* $\mathcal{O}$ *for* $m_j \in \omega$.

5 Definition of a truth function T

We define by recursion a function T which assigns elements of $\mathcal{O}$ to terms and truth values $\{\top, \bot\}$ to formulas of $\mathcal{L}$ dependently on the states $s \in \mathcal{S}$. Let t be a term without variables.

- If t is the name of an $a \in \mathcal{O}$, then $T(s,t) = a$ for all $s \in \mathcal{S}$.
- If t is not a name then it has the form $F_i(t_1 \ldots t_{n_i})$ since it is devoid of variables. Then $T(s, F_i(t_1 \ldots t_{n_i})) = f_i(s)(T(s,t_1), \ldots T(s,t_{n_i}))$

Let A be a closed formula of $\mathcal{L}$, and t_1, t_2 be terms. Then

- $T(s, t_1 = t_2) = \top$ iff $T(s,t_1) = T(s,t_2)$
- $T(s, P_j(t_1, \ldots t_{m_j})) = \top$ iff $(T(s,t_1), \ldots T(s,t_{m_j})) \in p_j(s)$
- $T(s, \neg\phi) = \top$ iff $T(s,\phi) = \bot$
- $T(s, \phi \vee \psi) = \top$ iff $T(s,\phi) = \top$ or $T(s,\psi) = \top$
- $T(s, \mathbf{G}\phi) = \top$ iff for all s' with s R s', $T(s',\phi) = \top$
  $T(s, \mathbf{H}\phi) = \top$ iff for all s' with s'R s, $T(s',\phi) = \top$
- $T(s, \exists x\phi) = \top$ iff there is an $a \in \mathcal{O}$ such that $T(s, \phi_x[a]) = \top$.

A formula ϕ is called valid in state $s \in \mathcal{S}$ of a temporal structure $\mathcal{M}$ iff $T(s,\phi') = \top$ for every $\mathcal{O}$-instance ϕ' of ϕ. This is denoted by $\mathcal{M}, \int \models \phi$. A formula A ϕ is called valid in a $\mathcal{K}_t$- structure $\mathcal{M}$ with the set of states $\mathcal{S}$, iff ϕ is valid in every $s \in \mathcal{S}$. We denote that by $\mathcal{M} \models \phi$. A formulaA ϕ is called $\mathcal{K}_t$- valid iff ϕ is valid in every $\mathcal{K}_t$ - structure. This is denoted by $\models_{\mathcal{K}_t} \phi$.

6 Soundness

Theorem 1 *If $\vdash_{\mathcal{K}_t} \phi$ then ϕ is $\mathcal{K}_t$- valid. Proof: AO is $\mathcal{K}_t$- valid, the definition of the truth function for the classical connectives being the same as in classical logic.*

A1. a: Assume there is a $\mathcal{K}_t$- structure A where $T(s, \mathbf{G}(\phi \to \psi) \to \mathbf{G}\phi \to \mathbf{G}\psi) = \bot$ for some state s. Then $T(s, \mathbf{G}(\phi \to \psi)) = T(s, \mathbf{G}\phi) = t$ and $T(s, \mathbf{G}\psi) = \bot$, i.e. $T(s', \phi \to \psi) = T(s',\phi) = t$ for all s' withsRs' and there is s'' with sRs'' such that $T(s", \psi) = \bot$. This is a contradiction, since from $T(s', \phi \to \psi) = T(s', \phi) = \top$ follows that $T(s',\psi) = \top$. 2.b. : assume there is a $\mathcal{K}_t$ - structure where $T(s, \phi \to \mathbf{H}\neg\mathbf{G}\neg\phi) = \bot$ for some state s. Then $T(s,\phi) = \top$ and there is a state $s' \in R(s)$ and $T(s', \neg\mathbf{G}\neg\phi) = \bot$. Hence $T(s', \mathbf{G}\neg\phi) = \top$ and therefore $T(s", \neg\phi) = \top$ for all states s'' such that $s' \in R(s'')$, i.e. $T(s'',\phi) = \bot$ for all states s'' with $s'' \in R(s')$. But this contradicts $T(s,\phi) = \top$ since $s \in R(s')$!

R0 is as in predicate logic and hence produces only valid formulae.

R1 a. If ϕ is a theorem of $\mathcal{K}_t$ then by induction assumption ϕ is $\mathcal{K}_t$ - valid, i.e. ϕ is true in every state of every $\mathcal{K}_t$- structure, hence $G\phi$ is true in every state of every $\mathcal{K}_t$- structure, i.e. $G\phi$ is a theorem of $\mathcal{K}_t$. Q.E.D.

7 Completeness

In this part we will show that every $\mathcal{K}_t$ - valid formula is a theorem of $\mathcal{K}_t$. Our proof is based on Henkin's completeness proof for classical logic [4] and is along the same lines as [5].

Definition 2 *A set of formulas, s, is called $\mathcal{K}_t$ -inconsistent if it has a finite subset $\{\phi_1, \phi_2, \ldots, \phi_k\}$ with $\vdash_{\mathcal{K}_t} \neg\phi_1 \vee \ldots \vee \neg\phi_k$. Otherwise s is called $\mathcal{K}_t$ - consistent (or consistent).*

Let s be a set of formulas. Then we denote by $V(s)$ the set of all terms occurring in formulas of s. We denote by $P(s)$ the set of all formulas of $\mathcal{L}(\mathcal{K}_t)$ containing only nonlogical symbols (terms, function and predicate symbols) of formulas of s.

Definition 3 *A set of formulas s is called complete (or $\mathcal{K}_t$ -complete) if*

- *s is $\mathcal{K}_t$ - consistent.*
- *s is maximal, i.e.for all ϕ in $P(s)$ holds: if $\phi \notin s$ then $s \cup \{\phi\}$ is inconsistent.*
- *s is saturated, i.e. for every existential formula $\exists x\phi \in s$ there is a formula $\phi_x[c] \in s$ for some constant $c \in V(s)$.*

Lemma 1 *Every consistent set of formulas can be extended to a complete set of formulas. Proof: analogous to that by Henkin-Hasenjäger.*

Let s be a consistent set of formulas and let a_1, a_2, ... be a sequence of 'new' object variables not in $V(s)$. We define $P^\star(s) = P(s) \cup \{j : j$ is a formula with variables from $a_1, a_2 \ldots\}$. Let $\{\exists x_i \phi_i(x_i)\}_{i=1,\ldots}$ be an enumeration of the existential formulas of $P^*(s)$. Then we form a set of formulas $s^\sim$, by adding to s a set of formulas $\exists x_i \phi_i(x_i) \rightarrow \phi(a_i)$ for every existential formula of $P(s)$. $s^\sim$ is consistent : assume for the contrary that there are formulas $\phi_1, \phi_2, \ldots, \phi_k$ in s, such that $\vdash_{\mathcal{K}_t} \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg(\neg\exists x_{i_0}\phi_{i_0}(x_{i_0}) \vee \phi_{i_0}(a_{i_0}))$

This gives
(1) $\vdash_{\mathcal{K}_t} \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \exists x_{i_0}\phi_{i_0}(x_{i_0})$ and
(2) $\vdash_{\mathcal{K}_t} \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg\phi_{i_0}(a_{i_0})$
From (2), we obtain
(3) $\vdash_{\mathcal{K}_t} \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \forall x\neg\phi_{i_0}(x)$ and hence
(4) $\vdash_{\mathcal{K}_t} \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg\exists x\phi_{i_0}(x)$
(1) and (4) imply $\vdash_{\mathcal{K}_t} \neg\phi_1 \vee \ldots \vee \neg\phi_k$, which contradicts the consistency of s.
$s^\sim$ is extended to $s^\star$ in the following way: let Θ be the set of all consistent formula sets, containing $s^\star$: $\Theta = \{s' : s \subset s'$ and s' consistent$\}$ Let H be a

chain in Θ, *i.e.* $H \subset \Theta$ *and* s_1, $s_2 \in H$ *implies* $s_1 \subset s_2$ *or* $s_2 \subset s_1$. *Then* $\bigcup H$ *is an upper bound of* H *in* Θ, *since* $s \subset \bigcup H$ *for all* $s \in H$. $\bigcup H$ *is consistent: if not, assume that there are formulas* $\phi_1, \phi_2, \ldots, \phi_k \in \bigcup H$ *such that* $\vdash_{\mathcal{K}_t} \neg\phi_1 \vee \ldots \vee \neg\phi_k$. *Then there are* $s_1, s_2, \ldots, s_k \in H$ *and* $\phi_i \in s_i$ *for* $1 \leq i \leq k$. *There is* $s_{k_0} \in \{s_1, s_2, \ldots, s_k\}$, *such that* $s_i \subset s_{k_0}$ *for* $1 \leq i \leq k$, *hence* $\phi_i \in s_{k_0}$ *for* $1 \leq i \leq k$, *which contradicts the consistency of* s_{k_0}. *Therefore* $\bigcup H \in \Theta$ *(since* $\bigcup H$ *is consistent).* $\bigcup H$ *is an upper bound of* H *and* $\bigcup H \in \Theta$. *By Zorn's Lemma* Θ *has a maximal element* $s^\star$, *i.e. for all* $s' \in \Theta$: $s^\star \subset s'$ *entails* $s' = s^\star$. $s^\star$ *is complete:*

- $s^\star$ *is consistent, because* $s^\star \in \Theta$
- $s^\star$ *is maximal: let* $\psi \notin s^\star$. *Then* $s^\star \cup \{\psi\} \notin \Theta$ *because* $s^\star$ *is a maximal element in* Θ. *Therefore, from the definition of* Θ, $s^\star \cup \{\psi\}$ *is not consistent.*
- $s^\star$ *is saturated: let* $\exists x\phi \in s^\star$. *From the definition of* $s^\sim$ *and because of* $s^\sim \subset s^\star$, *we have* $(\exists x\phi(x) \to \phi(a)) \in s^\star$. *Hence* $(\neg\exists x\phi(x)) \notin s^\star$ *and* $\neg(\exists x\phi(x) \to \phi(a)) \notin s^\star$ *because* $s^\star$ *is consistent. Assume now that* $\phi(a) \notin s^\star$. *Then* $s^\star \cup \{\phi(a)\}$ *is inconsistent, i.e. there are formulas* $\phi_1, \phi_2, \ldots, \phi_k \in s^\star$, *such that*

  (1) $\vdash_{\mathcal{K}_t} \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg\phi(a)$.

  By the same reasoning, there are formulas $\psi_1, \psi_2, \ldots, \psi_m \in s^\star$ *and* $\chi_1, \chi_2, \ldots, \chi_n \in s^\star$, *such that*

  (2) $\vdash_{\mathcal{K}_t} \neg\psi_1 \vee \ldots \vee \neg\psi_m \vee \neg\neg\exists x\phi$ *and*
  (3) $\vdash_{\mathcal{K}_t} \neg\chi_1 \vee \ldots \vee \neg\chi_n \vee \neg\neg(\exists x\phi(x) \to \phi(a))$.

  From (1) and (2) and (3) it follows easily

  (4) $\vdash \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg\psi_1 \vee \ldots \vee \neg\psi_m \vee \neg\chi\chi_1 \vee \ldots \vee \neg\chi\chi_n$

  which contradicts the consistency of $s^\star$.

Q.E.D.

Lemma 2 *Let* s *be a complete set of formulas. Then*

1. *If* $\phi \in s$ *then* $\neg\phi \notin s$
2. *If* $\phi \in P(s)$ *and* $\phi \notin s$*then*$\neg\phi \in s$
3. *If* $\phi \in s$ *and* $\psi \in P(s)$ *and* $\vdash \phi \to \psi$ *then* $\psi \in s$
4. *If* $\phi \vee \psi \in s$ *then* $\phi \in s$ *or* $\psi \in s$
5. *If* $\phi \vee \psi \in P(s)$ *and* $\phi \in s$ *or* $\psi \in s$ *then* $\phi \vee \psi \in s$

6. If $\phi[a] \in s$ then $\exists x\phi \in s$

7. If $\exists x\phi \in s$ then there is $a \in V(s)$ such that $\phi[a] \in s$

8. $t = t \in s$ for every term $t \in V(s)$

9. If $t_1 = t'_1 \in s, \ldots t_n = t'_n \in s$ and $F(t_1, ..., t_n) \in V(s)$ and $F(t'_1, t'_2, \ldots, t'_n) \in V(s)$ then $F(t_1, t_2, \ldots, t_n) = F(t'_1, t'_2, \ldots, t'_n) \in s$

10. If $t_1 = t'_1 \in s, \ldots t_n = t'_n \in s$ and $P(t_1, t_2, \ldots, t_n) \in s$ and $P(t'_1, t'_2, \ldots, t'_n) \in P(s)$ then $P(t'_1, t'_2, \ldots, t'_n) \in s$

Proof:

1. Since s is consistent and $\vdash \neg\phi \vee \neg\neg\phi$.

2. $s \cup \{\phi\}$ is inconsistent. Hence there are formulas $\phi_1, \phi_2, \ldots, \phi_k \in s$ and $\vdash_{\mathcal{K}_t} \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg\phi$. Assume $\neg\phi \notin s$. Then $s \cup \{\neg\phi\}$ is inconsistent and there are formulas $\psi_1, \psi_2, \ldots, \psi_k \in s$ such that $\vdash_{\mathcal{K}_t} \neg\psi_1 \vee \ldots \vee \neg\psi_m \vee \neg\neg\phi$. It follows that $\vdash_{\mathcal{K}_t} \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg\psi_1 \vee \ldots \vee \psi_m$, which contradicts the consistency of s.

3. is proven along the same lines as 2.

4. is proven along the same lines as 2

5. is proven along the same lines as 2

6. holds because s is saturated and from $\vdash_{\mathcal{K}_t} \phi[a] \to \exists x\phi[x]$ and from 3.

7. holds by the definition of complete set (saturated)

8. Assume for the contrary that $t = t \notin s$. Then there are formulas $\phi_1, \phi_2, \ldots, \phi_k \in s$ such that $\vdash_{\mathcal{K}_t} \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \neg(t = t)$. But $\vdash t = t$ by (identity axiom). Hence $\vdash_{\mathcal{K}_t} \neg\phi\phi_1 \vee \ldots \vee \neg\phi\phi_k$ from the cut rule contradicting the consistency of s.

9. Proof as in 7. by the equality axiom $t_1 = t'_1 \ldots t_n = t'_n \to F(t_1, t_2, \ldots, t_n) = F(t'_1, t'_2, \ldots, t'_n)$ of classical logic.

10. Proof as in 7. by the equality axiom $t_1 = t'_1 \ldots t_n = t'_n \to P(t_1, t_2, \ldots, t_n) \to P(t'_1, t'_2, \ldots, t'_n)$.

Q.E.D.

Definition 4 *Let s be a set of formulas. Then we define* $s^G := \{\phi : \mathbf{G}\phi \in s\}$ $s^H = \{\phi : \mathbf{H}\phi \in s\}$

Definition 5 *A system of sets $\mathcal{S}$ is called complete if*

1. *Every element s of S is a complete set of formulas.*

2. *If $s \in S$,$\phi \in P(s)$ and $s^G \cup \{\phi\}$ is consistent, then there is $s' \in S$ such that $s^G \cup \{\phi\} \subset s'$*

Lemma 3 *For every complete set of formulas s, there is a complete system of sets S with $s \in S$.*
Proof: For every $\phi \in P(s)$ such that $s^G \cup \{\phi\}$ is consistent, we extend $s^G \cup \{\phi\}$ to a complete set for formulas s', which exists by 1. So we form systems of sets S_0, S_1,... by
$S_0 := \{s\}$
$S_{i+1} := \{x : x$ *is a complete extension of* $y^G \cup \{\phi\}$ *where* $y \in S_i$ *and* $\phi \in P(y)\}$
$S := \bigcup\{S_i : i \in \omega\}$ Q.E.D.

Definition 6 *Given a complete system of sets S, we define two binary relations R^G and R^H on S : 1. $sR^G s'$ iff_{def} $s^G \subseteq s'$ 2. $sR^H s'$ iff_{def} $s^H \subseteq s'$*

Lemma 4 *$sR^G s'$ iff $s' R^H s$*
Proof: Let be $sR^G s'$ and $\phi \in s'^H$. We shall show that $\phi \in s$. $\phi \in s'^H$ iff $\mathbf{H}\phi \in s'$ by construction of s'^H from s' (Definition 4) iff $\neg\mathbf{H}\phi \notin s'$ by completeness of s' and Lemma 2,1 and 2. Hence $\neg\mathbf{H}\phi \notin s^G$ by assumption $s^G \subseteq s'$ iff $G\neg\mathbf{H}\phi \notin s$ by construction of s^G from s (Definition 6) iff $\neg G\neg\mathbf{H}\phi \in s$ by completeness of s and Lemma 2,1 and 2. Hence $\phi \in s$ by Lemma 2,3 and T3.a.
'$\Leftarrow$' analogously. Q.E.D.

Lemma 4 shows that R^H is the inverse relation of R^G. We therefore define $R = R^G$.

Lemma 5 *Let S be a complete system of sets, let $s \in S$. Then*

a $\mathbf{G}\phi \in s$ *iff* $\phi \in s'$ *for every* $s' \in R(s)$

b $\mathbf{H}\phi \in s$ *iff* $\phi \in s'$ *for every* $s \in R(s')$

Proof: of a. : ($\Rightarrow$): $\mathbf{G}\phi \in s$ iff $\phi \in s^G$ and therefore $\phi \in s'$ for all $s' \in R(s)$ by the definition of R. ($\Leftarrow$) : We show first that $s^G \cup \{\neg\phi\}$ is inconsistent. Assume that $s^G \cup \{\neg\phi\}$ is consistent. Then there exists $s' \in S$ such that $s^G \cup \{\neg\phi\} \subseteq s'$ from Definition 5. Hence $s^G \subseteq s'$ and therefore $s' \in R(s)$. $\phi \in s'$ from the precondition and therefore $\neg\phi \notin s'$ by Lemma 2,1 (since s' is complete). This contradicts $s^G \cup \{\neg\phi\} \subseteq s'$! Hence $s^G \cup \{\neg\phi\}$ is inconsistent. So there exist formulas $\phi_1, \phi_2, \ldots, \phi_k \in s^G$ such that $\vdash_{\mathcal{K}_t} \neg\phi_1 \vee \ldots \vee \neg\phi_k \vee \phi \vdash \phi_1 \wedge \ldots \wedge \phi_k \rightarrow \phi \vdash \mathbf{G}(\phi_1 \wedge \ldots \wedge \phi_k) \rightarrow \mathbf{G}\phi$ by D2 a) $\vdash \mathbf{G}\phi_1 \wedge \ldots \wedge \mathbf{G}\phi_k \rightarrow \mathbf{G}\phi$ by T3 $\vdash_{\mathcal{K}_t} \neg\mathbf{G}\phi_1 \vee \ldots \vee \neg\mathbf{G}\phi_k \vee \mathbf{G}\phi$

$\mathbf{G}\phi_1, \mathbf{G}\phi_2, \ldots, \mathbf{G}\phi_k \in s$, therefore $\neg\mathbf{G}\phi \notin s$, since s is consistent and hence $\mathbf{G}\phi \in s$ by Lemma 2,2 and the completeness of s. The proof of b is analogous.
Q.E.D.

We are now in a position to define a temporal (reflexive) model. The objects will be the equivalence classes according to the identity of terms, $=$.

Definition 7 *Let S be a system of sets. Let t_0, $t_1 \in V(s)$ for some $s \in S$. Then $t_0 \cong t_1 iff_{def} (t_0 = t_1) \in s \forall s \in S$.*

Lemma 6 *$\cong$ is an equivalence relation. Proof::*

a) $\cong$ is reflexive by 2,9.

b) $\cong$ is symmetric by $\vdash t = t' \wedge t = t" \to t' = t"$ equality axiom. $\vdash t = t' \wedge t = t \to t' = t$ replace $t"$ by t. Since $t = t \in s$ by Lemma 2.9., we get $t = t' \in s$ entails $t' = t \in s$ by Lemma 2.11.

c) $\cong$ is transitive $\vdash t' = t" \wedge t = t' \to t = t"$ $t' = t" \in s$ and $t = t' \in s$ entails $t = t" \in s$ by Lemma 2,3.

Q.E.D.

We denote by $[t]$ the $\cong$-equivalence class of t.

We define a temporal structure $(\{\phi_s : s \in S\}, R)$, where

(1) S is the system of sets in Definition 5.

(2) R is the relation in Definition 6.

(3) $A_s = (O, (f_i(s)), (p_j(s)), =)$ is a classical structure.

(3.1) O is the set of $\cong$ - equivalence classes.
(3.2) $f_i^s([t_1], [t_2], \ldots, [t_{n_i}]) = [F_i(t_1, t_2, \ldots, t_{n_i})]$
(3.3) $([t_1], [t_2], \ldots, [t_{m_j}]) \in p_j^s$ iff $P_j(t_1, t_2, \ldots, t_{m_j}) \in s$
(3.4) $=$ is the equality of the equivalence classes.

The definition of the functions f_i and the predicates p_j depends only on the $\cong$ - classes and not on the terms, i.e.

$[t_0] = [t'_0] \ldots$ and $[tn_i] = [t'n_i]$ entails

$f_i^s([t_1], [t_2], \ldots, [t_{n_i}]) = f_i^s([t'_1], [t'_2], \ldots, [t'_{n_i}])$ by Lemma 2, 10.

$[t_1] = [t'_1]$,... and $[t_{m_j}] = [t'_{m_j}]$ entails

$([t_1], [t_2], \ldots, [t_{m_j}]) \in p^s_j)$ iff $([t'_1], [t'_2], \ldots, [t'_{m_j}]) \in p_j(s)$ by Lemma 2,11.

Definition of the truth value function T:

For every term t we set $T(s, t) = [t]$. The truth value for a formula is given inductively by the definition of functions and predicates (3).

Lemma 7 *For every closed formula $\phi \in P(s)$ is $T(s, \phi) = \top$ iff $\phi \in s$. Proof: by induction over formulas:*

1. *$T(s, t = t') = \top$ iff $T(s, t) = T(s, t')$ iff $[t] = [t']$ iff $\forall s \in S(t = t') \in s$ by definition of $\cong$.*
2. *$T(s, P_j(t_1, t_2, \ldots, t_{m_j})) = \top$ if $(T(s, t_1), \ldots T(s, t_{m_j})) \in p^s_j$ iff $([t_1], [t_2], \ldots, [t_{m_j}]) \in p_j(s)$ if $p_j(t_1, t_2, \ldots, t_{m_j}) \in s$.*
3. *$T(s, \neg\psi) = \top$ iff $T(s, \psi) = \bot$ if $\psi \notin s$ by induction hypothesis iff $\neg\psi \notin s$ from Lemma 2,1. and 2.*
4. *$T(s, \psi \vee \chi) = t$ if $T(s, \psi) = \top$ or $T(s, \chi) = \top$ iff $\psi \in s$ or $\chi \in s$ by induction hypothesis if $\psi \vee \chi \in s$ by Lemma 2, 4. and 5.*
5. *$T(s, \exists x\psi) = \top$ if there is $c \in O$ such that $T(s, \psi_x[c]) = t$ if $\psi_x[c] \in s$ by induction hypothesis iff $\exists x\psi \in s$ (from Lemma 2,6 and 7.*
6. *$T(s, \mathbf{G}\psi) = \top$ if for all $s' \in R(s')$, $T(s', \psi) = \top$ iff for every $s' \in R(s')$, $\psi \in s'$ by induction hypothesis iff $\mathbf{G}\psi \in s$ by Lemma 6.*
7. *$T(s, H\psi) = \top$ if for all $s' such that s \in R(s')$, $T(s', \psi) = \top$ iff for all $s' : sRs'\psi \in s'$ by induction hypothesis iff $H\psi \in s$ from Lemma 6.*

Q.E.D.

Theorem 2 *Theorem 2. Completeness theorem. Every $\mathcal{K}_t$- valid formula is deducible in $\mathcal{K}_t$. Proof:: Assume for the contrary that there is a valid formula ϕ which is not deducible in $\mathcal{K}_t$. Then, by definition 2, $\{\neg\phi\}$ is a $\mathcal{K}_t$- consistent set of formulae. $\{\neg\phi\}$ can be extended to a complete set of formulae s, by Lemma 1, where $\neg\phi \in s$. By Lemma 3 there is a complete system of sets S with $s \in S$. By Lemma 7, $T(s, \neg\phi) = \top$ and hence $T(s, \phi) = \bot$, which contradicts the validity of ϕ.* Q.E.D.

8 Representability of temporal properties

As mentioned in the beginning of this chapter, the axioms of $\mathcal{K}_t$ describe a "minimal temporal system", the temporal relation R does not have any particular property. Hence these axioms are true for any temporal structure. But in order to decide if some temporal system is useful for a given application we have to look for its capacities to describe the structures we have to describe. It is important to understand that the representability concerns problems related to the axiomatization of temporal logic, given a temporal language with modal temporal operators and a notion of semantics as presented above, based on temporal structures formulated as Kripke-type structures.

8.1 Transitive time

A very natural property time should have is the transitivity. If a formula ϕ is true in all the future (past) then it is true in all the future (past) that ϕ is true in the future.

Axiom 4 *(a)* $\mathbf{G}\phi \to \mathbf{GG}\phi$ *(b)* $\mathbf{H}\phi \to \mathbf{HH}\phi$

This axiom is only true for temporal structures where the temporal relation R is transitive.

Corollary 1 *The class of transitive structures can be expressed in temporal logic.*

Proof: The axiom 4 characterizes this class. We show first that the axiom 4 is valid in every transitive structure. Suppose for the contrary that there is a transitive structure $M_0 = (\{\mathcal{A}_s : s \in \mathcal{S}\}, R)$ *in which axiom 4 is not valid.Then there is a state* $s_0 \in \mathcal{S}$ *such that* $T(s_0, \mathbf{G}\phi \to \mathbf{GG}\phi) = \bot$. *Hence,* $T(s_0, \mathbf{G}\phi) = \top$ *and* $T(s_0, \mathbf{GG}\phi) = \bot$, *i.e. for all* $s' \in R(s_0)$, $T(s_0, \phi) = \top$ *and there is* $s'' \in R(s')$ *and there is* $s''' \in R(s'')$ *and* $T(s_0, \phi) = \bot$. *But by transitivity of R, we have also* $s''' \in R(s_0)$. *Contradicton!*
On the other hand, axiom 4 is not valid in non-transitive structures: Let $M_0 = (\{\mathcal{A}_s : s \in \mathcal{S}\}, R)$ *be a structure where R is not transitive. Then there is* t, t', $t'' \in \mathcal{S}$ *and* $t' \in Rt(t)$, $t'' \in Rt(t')$ *but not* $t'' \notin Rt(t)$. *Let* $T(t', \phi) = \top$ *for every* $s' \in R(s)$ *and* $T(t'', \phi) = \bot$. *Then* $T(s, \mathbf{G}\phi) = \top$ *and* $T(s, \mathbf{GG}\phi) = \bot$. *Therefore* $T(t, \mathbf{G}\phi \to \mathbf{GG}\phi) = \bot$. Q.E.D.

8.2 Reflexive time

Axiom 5 *(a)* $\mathbf{G}\phi \to \phi$ *(b)* $\mathbf{H}\phi \to \phi$

This axiom is only true for temporal structures where the temporal relation is reflexive.

Corollary 2 *The class of reflexive structures can be expressed in temporal logic.*

Proof: Axiom 5 characterizes this class. We show first that axiom 5 is valid in every reflexive structure. Suppose for the contrary that there is a reflexive structure $M_0 = (\{\mathcal{A}_s : s \in \mathcal{S}\}, R)$ in which axiom 5 is not valid. Then there is $s \in \mathcal{S}$ such that $T(s, \mathbf{G}\phi) = \top$ and $T(s, \phi) = \bot$. This is the case when for all $s' \in R(s)$, $T(s, \phi) = \top$. But by reflexivity, we have $s \in R(s)$, and $T(s, \phi) = \bot$, contradiction!
On the other hand, axiom 5 is not valid in nonreflexive structures: Let be $M_0 = (\{\mathcal{A}_s : s \in \mathcal{S}\}, R)$ a structure where R is not reflexive. Then there is $s \in \mathcal{S}$ and $s \notin R(s)$. Let us assign $T(s', \phi) = \top$ for all $s' \in R(s)$, hence $T(s', \mathbf{G}\phi) = \top$ and let be $T(s, \phi) = \bot$. Q.E.D.

8.3 Symmetric time

Axiom 6 *(a)* $\mathbf{FG}\phi \to \phi$ *(b)* $P\mathbf{H}\phi \to \phi$

This axiom is only true for temporal structures where the temporal relation is symmetric.

Corollary 3 *The class of symmetric structures can be expressed in temporal logic.*

Proof: Axiom 6 characterizes this class. We show first that axiom 6 is valid in every symmetric structure. Suppose for the contrary that there is a symmetric structure $M_0 = (\{\mathcal{A}_s : s \in \mathcal{S}\}, R)$ in which axiom 6 is not valid. Then there is $s \in \mathcal{S}$ such that $T(s, \mathbf{FG}\phi) = \top$ and $T(s, \phi) = \bot$, this is the case when $T(s, \mathbf{G}\neg\mathbf{G}\phi) = \bot$, i.e. there is $s \in R(s')$ and $T(s', \neg\mathbf{G}\phi) = \bot$, i.e. $T(s', \mathbf{G}\phi) = \bot$, therefore for all $s'' \in R(s')$, $T(s'', \phi) = \bot$. But by symmetry, we have also $s \in R(s')$ (since $t \in R(s)$, and $T(s, \phi) = \bot$, contradiction! On the other hand, axiom 6 is not valid in nonsymmetric structures: Let $M_0 = (\{\phi_s : s \in \mathcal{S}\}, R)$ a structure where R is not symmetric. Then there are s, $s' \in \mathcal{S}$ and $s' \in R(s)$ but not $s \in R(s')$. Let us assign $T(s', \phi) = \top$ for all s' such that $s' \in R(s)$, hence $T(s', \mathbf{G}\phi) = \top$ and let be $T(s, \phi) = \bot$.
Q.E.D.

8.4 Linear time

Axiom 7 *(a)* $\mathbf{G}(\phi \vee \psi) \wedge \mathbf{G}(\phi \vee \mathbf{G}\psi) \wedge \mathbf{G}(\mathbf{G}\phi \vee \psi) \to \mathbf{G}\phi \vee \mathbf{G}\psi$
(b) $\mathbf{H}(\phi \vee \psi) \wedge \mathbf{H}(\phi \vee \mathbf{H}\psi) \wedge \mathbf{H}(\mathbf{H}\phi \vee \psi) \to \mathbf{H}\phi \vee \mathbf{H}\psi$

This axiom is only true for temporal structures where the temporal relation is linear.

Corollary 4 *The class of linear structures can be expressed in temporal logic.*

Proof: Axiom 7 characterizes this class. We show first that axiom 7 is valid in every linear structure. Suppose for the contrary that there is a linear structure $M = (\{\mathcal{A}_s : s \in \mathcal{S}\}, R)$ in which axiom 7 is not valid. Then there is $s \in \mathcal{S}$ such that $T(s, \mathbf{G}(\phi \vee \psi)) = \top$ and $T(s, \mathbf{G}(\phi \vee \mathbf{G}\psi)) = \top$ and $T(s, \mathbf{G}(\mathbf{G}\phi \vee \psi)) = \top$ and $T(s, \mathbf{G}\phi \vee \mathbf{G}\psi) = \bot$, this is the case when for all $s' \in R(s)$, $T(s', (\phi \vee \psi)) = \top$, and ($T(s', \phi) = \top$ or for all $s'' \in R(s')$, $T(s'', \psi) = \top$) and ($T(s', \psi) = \top$ or $T(s'', \phi) = \top$) and there is $s''' \in R(s)$ and $T(s''', \phi) = \bot$, and there is $s'''' \in R(s)$ and $T(s'''', \psi) = \bot$. But then, we have $T(s''', \phi) = \bot$ and $T(s''', \psi) = \top$ and $T(s''', \mathbf{G}\psi) = \top$ and $T(s'''', \psi) = \bot$ and $T(s'''', \phi) = \top$ and $T(s'''', \mathbf{G}\phi) = \top$. By linearity and since $s''' \neq s''''$, we have either $s'''' \in R(s''')$ or $s''' \in R(s'''')$. But then in the first case, ψ cannot be false in s'''' (since $\mathbf{G}\psi$ is true in s''') and in the second case ϕ cannot be false in s''' (since $\mathbf{G}\phi$ is true in s''''), contradiction! On the other hand, axiom 7 is not valid in nonlinear structures : Consider a structure where R is not linear. Then there are s, s', $s'' \in \mathcal{S}$ and $s' \in R(s)$ and $s'' \in R(s)$ and $s \neq s'$ and $s' \notin R(s)$ and $s \notin R(s')$. Let us assign $T_0(s', \phi) = \bot$ and $T_0(s'', \psi) = \bot$, and $T_0(s', \psi) = \top$ and $T_0(s'', \phi) = \top$,hence $T_0(s, \mathbf{G}\phi) = \bot$ and $T_0(s, \mathbf{G}\psi) = \bot$. But $T_0(s, \mathbf{G}(\phi \vee \mathbf{G}\psi)) = \top$ and $T_0(s, \mathbf{G}(\mathbf{G}\phi \vee \psi)) = \top$. and $T_0(s, \mathbf{G}(\phi \vee \psi)) = \top$. Hence T_0 falsifies axiom 7 Q.E.D.

8.5 Dense time

A temporal structure is dense if R is dense; a relation R on a set $\mathcal{S}$ is dense if for all s, $s' \in \mathcal{S}$ such that $s' \in R(s)$ there is $s'' \in R(s)$ and $s' \in R(s'')$.

Axiom 8 *(a)* $\mathbf{GG}\phi \rightarrow \mathbf{G}\phi$ *(b)* $\mathbf{HH}\phi \rightarrow \mathbf{H}\phi$

This axiom is only true for temporal structures where the temporal relation is dense.

Corollary 5 *The class of dense structures can be expressed in temporal logic.*

Proof: Axiom 8 characterizes this class. We show first that axiom 8 is valid in every dense structure. Suppose for the contrary that there is a dense structure $M = (\{\mathcal{A}_s : s \in \mathcal{S}\}, R)$ in which axiom 8 is not valid. Then there is $s \in \mathcal{S}$ such that $T(s, \mathbf{GG}\phi) = \top$ and $T(s, \mathbf{G}\phi) = \bot$. This is the case when for all $s' \in R(s)$, $T(s', \mathbf{G}\phi) = \top$, and there is $s'' \in R(s)$ and $T(s'', \phi) = \bot$. But since R is dense, there is $s''' \in R(s)$ and since $T(s''', \mathbf{G}\phi) = \top$, this *is a contradiction! On the other hand, axiom 8 is not valid in nondense structures: Consider a structure where R is not dense. Then there are s, $s' \in \mathcal{S}$ and $s' \in R(s)$ and there is no $s'' \in R(s)$ such that $s' \in R(s'')$. Let us assign $T_0(s', \phi) = \bot$ and $T_0(s, \mathbf{GG}\phi) = \top$, i.e. $T_0(s', \mathbf{G}\phi) = \top$. Clearly, T_0 falsifies the axiom 8.* Q.E.D.

Remark 1 *Every reflexive structure is also dense and the axiom 8 can be derived from the axiom 5. As already mentioned above, there are classes of temporal structures which cannot be expressed by temporal logic.*

Corollary 6 *The class of not-symmetric structures cannot be expressed by temporal logic.*

Proof: Suppose there is an axiom ϕ which characterizes this class. But then ϕ is false in the structure $M = (\{\mathcal{A}_s\}, \{(s,s)\})$, since this is a symmetric structure. But then ϕ is also false in $M' = (\{\mathcal{A}_s, \mathcal{A}'_s, \mathcal{A}''_s\}, \{(s,s), (s', s'')\})$, which is not symmetric, hence ϕ does not characterize the class of symmetric structures (we have found a structure, which is not symmetric and where ϕ is true). Q.E.D.

From corollaries 7 and 3, we obtain straightforwardly

Corollary 7 *The class of structures which can be expressed in temporal logic is not closed under complementation.*

Corollary 8 *The class of structures expressible in temporal logic is closed under intersection.*

Proof: If ϕ expresses the class $\mathcal{C}_A$ and ψ expresses the class $\mathcal{C}_B$, then $\phi \wedge \psi$ expresses the class $\mathcal{C}_A \cap \mathcal{C}_B$. Q.E.D.

Corollary 9 *The class of connected structures cannot be expressed in temporal logic. Proof: Suppose ϕ characterizes this class. Then ϕ is false in $M = (\{A_s, A'_s\}, \{(s,s), (s',s')\})$. But then either $T(s,\phi) = \bot$ or $T(s', \phi) = \bot$. Hence ϕ is false either in $(\{A_s\}, \{(s,s)\})$ or in $(\{A'_s\}, \{(s',s')\})$ which are both connected structures, hence ϕ cannot characterize this class.* Q.E.D.

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